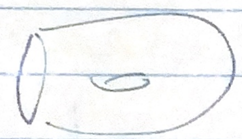
Sep 18 Boundary expansion and singular Yamabe compactification in Lovelock gravity.

I. Conformal geometry.



X^{n+1} manifold with boundary

g interior metric

$x \geq 0$ bdf. $x = 0 \iff$ on ∂X

$dx \neq 0$ $\qquad \overset{\shortparallel}{M}$.

g is conformally compact if

$$\bar{g} = x^2 \cancel{g}\, g \text{ extends smoothly to } \bar{X}.$$

a metric on.

Choice of x is not unique. differ by e^{φ}.

a special choice is such that $|dx|_{\bar{g}} = 1$

represents sectional curvature as $x \to 0$

Near the boundary $M \times [0, \varepsilon)$

$$g = \frac{dx^2 + h_x}{x^2}$$

where h_x is a 1-parameter family of metric on M.

~~all~~ but it gives a well-defined conformal class $[h] = [\bar{g}|_{TM}]$.

Prototype example: hyperbolic ball. B^{n+1}

$$g = \frac{4}{(1-r^2)^2} dy^2. \qquad y \in B^{n+1}$$

bdf. $x = \frac{1-r^2}{2}$ $\qquad [h] = [g_{S^n}]$.

⊛ AdS/CFT

II. Lovelock tensors

Einstein's theory + higher order curvature corrections.
sensitive to the dimension
away from ghosting

Einstein tensor $E = Ric + \kappa g$.
symmetric, divergence free,
polynomial in $g, \partial g, \partial^2 g$.
linear in $\partial^2 g$

solution space spanned by Ric, g.

allow nonlinearity

larger solution space spanned by $Ric^{(2q)}, g$

Def. $Ric^{(2q)}{}^{j}_{i} = \boxed{\delta^{i\,i_1 \cdots i_{2q}}_{j\,j_1 \cdots j_{2q}}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2q-1} i_{2q}}{}^{j_{2q-1} j_{2q}}$

generalized Kronecker delta
forces us to contract distinct pairs.

$$Ric^{(2q)} = \underbrace{c_g^{2q-1}}_{(2q-1)\text{-pairs chosen}} (Rm^q)$$

$scal^{(2q)} = tr(Rm^{(2q)}) = c_g^{(2q)}(Rm^q)$.

Examples. $q=1$ Einstein
$q=2$ Gauss-Bonnet term.

Rmk. $2q > n+1$ not enough distinct pairs.
$Ric^{(2q)}$ collapse to zero.

$2q = n+1$ critical in the sense that $\int \mathrm{scal}^{(2q)}$ is topological (proportional to $\chi(X)$)

This explains why in dim $= 4$, Lovelock theory is nothing more than Einstein theory.

Gauss-Bonnet theorem:

$$\frac{1}{8\pi^2}\int \mathrm{scal}^2 - 4|\mathrm{Ric}|^2 + |\mathrm{Rm}|^2 = \chi(X).$$

<u>Def.</u> Lovelock tensor

$$F_g(\alpha) = \sum \alpha_q\left(\mathrm{Ric}^{(2q)} - \lambda^{(2q)} g\right) - \frac{\alpha_q}{2q}\left(\mathrm{scal}^{2q} - (n+1)\lambda^{(2q)}\right) g$$

α_q coupling constants.

$\lambda^{(2q)}$ normalization.

g is a Lovelock metric if $F_g(\alpha) = 0$

Variational property: Lovelock eqn. is the Euler-Lagrange eqn of the total $\mathrm{scal}^{(2q)}$ curvature.

III. Boundary asymptotics.

Two types of questions.

1. tensorial Recall $g = \frac{dx^2 + h_x}{x^2}$

$$h_x = \begin{cases} h_0 + h_2 x^2 + \boxed{(\text{even}) +} \cdots + h_{n-1} x^{n-1} + h_n x^n & n \text{ odd} \\ h_0 + h_2 x^2 + \boxed{(\text{even}) +} \cdots + h_{n,1} x^n \log x + h_n x^n & \text{even} \end{cases}$$

Fefferman–Graham expansion.

2. scalar. A choice of bdf u s.t. $\text{scal}^{(2q)}_{u^{-2}g} = \text{const}.$

$$u = u_0 + u_1 x^1 + \cdots + u_{n+2,1} x^n \log x + u^{n+2} x^{n+2} + \cdots$$

In the first case, only even terms shows up until order n.

In the second case, no such property.

In both case, $\log x$ pops up: a direct computation we use $E = \text{Ric} + ng$ as an example.

Near M. E decompose into tangential, mixed and normal components.

To get expansion of h_x. use tangential component $E_{ij} \rightsquigarrow$ gives an ODE

$$x h''_{ij} + (1-n) h'_{ij} - \text{tr}(h') h_{ij} - x h^{kl} h'_{ik} h'_{jl}$$
$$+ \frac{x}{2} \text{tr}(h') h'_{ij} - 2x \,\text{Ric}(h_x) = 0$$

Parity $\Rightarrow$ $h'|_M = 0$.

differentiating this ~~(s+1)~~ $(s-1)$-times $\Rightarrow$

$$(s-n) \partial_x^s h_{ij} - h^{kl} (\partial_x^s h_{kl}) h_{ij} = \text{l.o. derivatives}$$

Aside: similar arguement for $\mathrm{scal}^{(g)} = \mathrm{const}$.
when we replace $E=0$ with conf change of $\mathrm{scal}^{(2g)}$

induction stops at $s=n$. where we introduce $\log x$ term to continue.

<u>Def.</u> Ambient obstruction to a smooth expansion of h_x. is given by $h_{n,1}$ (log term coeff).

Related question we may ask.

1. Is the formal FG expansion attained by conf. compact Lovelock metrics?
2. What do we know about the obstruction tensor?

These were studied for Einstein condition, we use similar approach. Useful tool here is the DeTurck's gauge fixing method. (remove terms that obstructing ellipticity) (due to diffeomorphism invariance).

Define modified Lovelock tensor

$$Q_\alpha(g,t) = F_\alpha(g) - \underbrace{\Phi_\alpha(g,t)}_{(c_1\delta_g^* + \underbrace{c_2 g\delta_g}_{\text{new term, due to } \mathrm{scal}^{(2g)}})(g t^{-1} B_g(t))}$$

$$(D_1 Q_\alpha)(g,t)(F) \quad \frac{A_1(\alpha)}{4}\Big(-(n-1)(\Delta_g + 2n)(\ell_{g_0}(r)\, g_0) + 2(\Delta_{g_0} - 2)\, r_0\Big) + O(x^{N+1})$$

at asymptotically hyperbolic g_0.

Elliptic regularity and isomorphism away from indicial roots of the Laplace type operator near M allows us to conclude

- existence of Lovelock metric on $\mathbb{B}^{n+1}$, where the conformal infinity h is $C^{2,\alpha}$ closed to g_{S^n}.
- Lovelock metric + conformally compactness
  $\Rightarrow$ polyhomogeneigty $\Rightarrow$ FG expansion attained

Application of boundary asymptotics.

1. well-defined renormalized integral.

$$\mathrm{vol}\,(x > \varepsilon) = \begin{cases} c_0 \varepsilon^{-n} + c_2 \varepsilon^{-n+2} + \cdots + c_{n-1} \varepsilon^{-1} + V + o(1) & n \text{ odd} \\ c_0 \varepsilon^{-n} + c_2 \varepsilon^{-n+2} + \cdots + c_{n-2} \varepsilon^{-2} + \boxed{L} \log(\frac{1}{\varepsilon}) + V + o(1) & \text{even} \end{cases}$$

$$L \propto \int_M Q. \quad Q\text{-curvature}$$

2. Scattering problem $s > n/2$

$$\begin{cases} \Delta_g - s(n-s)\, u = 0 \\ u = F x^{n-s} + G x^{s} \end{cases} \qquad F, G \in C^\infty(X).$$

Leading order term of obstruction tensor.
a direct computation. extracting log coeff.
take $F_\alpha(g)$. obstruction tensor is given by

$$\mathcal{O} = x^{2-n}\, tf(F_\alpha(g))\big|_{x=0}$$

We compute

$$\partial_x^s\big|_{x=0} x\left(Ric^{(2q)} - \lambda^{(2q)} g_{ij}\right)$$

$$= \frac{A_1(\alpha)}{2}\left[(n-s-1)\,\partial_x^{s+1}\big|_{x=0} h + 2s\,\partial_x^{s-1}\big|_{x=0} Ric_h\right]$$

$+$ trace terms $+$ l.o. derivatives

Same recurrence relation up to a coeff $\frac{A_1(\alpha)}{2}$.

$$\mathcal{O} = \frac{A_1(\alpha)}{2n}\, \Delta^{n/2-2}\left(P_{ij,k}{}^{k} - P_k{}^{k}{}_{,ij}\right)$$

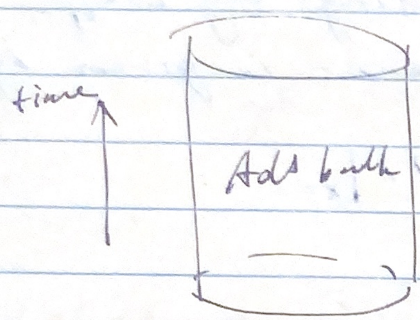
In the Einstein case, it is known that if h is conformal to an Einstein metric, then $\mathcal{O} = 0$.
in fact

if $Ric_h = 4\lambda(n-1)h$

then $g = \dfrac{dx^2 + (1-\lambda x^2)^2 h}{x^2}$

In Lovelock, with $q \le 2$, same conformal factor $f = (1-\lambda x^2)^2$ holds only for a particular choice of coupling constants, which corresponds to the case $A_1(\alpha) = 0$

AdS / CFT correspondance.



X^{n+1}, gravitational theory

CFT boundary M^n, conformal field theory

Riemannian geometry on X "easy"

$\updownarrow$

conformal geometry on M "hard".

Fefferman - Graham expansion is a relization of this duality.